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A series solution for the effective properties of incompressible viscoelastic media

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ABSTRACT

This paper presents a series solution for the homogenization problem of a linear viscoelastic periodic incompressible composite. The method uses the Laplace transform and the correspondence principle which are combined with the classical expansion along Neumann series of the solution of the periodic elasticity problem in Fourier space. The terms of the Neumann series appear as decoupled, containing geometry dependent terms and viscoelastic properties dependent terms which are polynomial fractions whose inverse Laplace transforms are provided explicitly.

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1. Introduction

The methods used for predicting the effective properties of heterogeneous viscoelastic composites comprise solutions to the problem of the complex moduli (Brinson and Lin, 1998) with applications to dynamic problems, but the most important practical problem is to predict relaxation or creep functions. This last objective is generally attained by using Laplace transform of the equations, the main problem being to produce accurately the inverse Laplace transform. This was effected for Mori–Tanaka or Self-Consistent modelings (Beurthey et al., 2000; Rougier et al., 1994; Le et al., 2007), which allow to obtain the relaxation function explicitly or by using a simple 1D integral. In addition, all effective behaviours must comply with some asymptotic conditions, as obtained for example in the case of Maxwell constituents (Suquet, 2012).

The determination of the effective properties of periodic media using the classical Neumann series was used from a theoretical point of view since a long time (Brown, 1955), for conductivity, or for elasticity, by using the related Green's tensors. The practical application in conduction and elasticity rests on iterative schemes and on the use of the Fourier transform because the Fourier transform of the Green's tensor is known explicitly for an homogeneous medium in the case of elastic constitutive equations (Michel et al., 1999, 2001; Monchiet and Bonnet, 2012; Moulinec and Suquet, 2003, 1994, 1998; Bonnet, 2007). Approximate solutions based on Nemat-Nasser et al. (1982), which use Fourier transforms of the solutions, can produce explicit results in the case of viscoelastic components (Luciano and Barbero, 1994; Barbero and Luciano, 1995; Hoang-Duc et al., 2013) but these solutions are no more valid for high concentrations of inclusions or high contrasts. Accurate solutions at any concentration were obtained either by time-step integration (Lahellec and Suquet, 2007) or by numerical Laplace inverse, generally using collocation methods (Yi et al., 1998). However, a method based on Fourier transform, but which does not need numerical time-step integration or numerical Laplace inversion would be highly desirable. This is the aim of the paper.

In the following, this method will be called "NS method". The solution for determining the macroscopic behavior of viscoelastic periodic media is developed by using the classical Neumann Series for the effective elastic properties.

The paper is organized as follows: The constitutive relation used for the individual constituents is presented in Section 2. Then, we present in Section 3 simplified formulations of the effective properties of composite elastic media made of isotropic constituents with a decoupling of elastic properties and geometry properties in each term of the Neumann series. This decoupling appears only in some specific cases, including the case of incompressible constituents. This result is used in the next section to determine the expression of the relaxation function of the viscoelastic periodic composites at the macroscale. Finally, the method is checked against results coming from previous works.

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2. Linear viscoelastic behavior

2.1. Constitutive equations for an isotropic viscoelastic medium

In the following, a composite material is studied where the constituting phases are either elastic or non ageing viscoelastic. The constitutive stress–strain relation of a non-ageing viscoelastic material is given classically (Christensen, 1969; Salençon, 2009), by a Stieltjes integral as:

$$\boldsymbol{\sigma}(t) = \int_0^t \mathbb{R}(t-\tau) : \frac{d\boldsymbol{\epsilon}(\tau)}{d\tau} d\tau = \mathbb{R}(: \circledast)\dot{\boldsymbol{\epsilon}} \tag{1}$$

or reversely:

$$\boldsymbol{\epsilon}(t) = \int_0^t \mathbb{J}(t-\tau) : \frac{d\boldsymbol{\sigma}(\tau)}{d\tau} d\tau = \mathbb{J}(: \circledast)\dot{\boldsymbol{\sigma}} \tag{2}$$

where $\mathbb{R}$, $\mathbb{J}$ are tensorial relaxation and creep functions. The dot denotes the time derivative and the convolution of two functions f and g, denoted as "$f \circledast g$", is defined by:

$$(f \circledast g)(x) = \int_{-\infty}^{+\infty} f(x-t)g(t)dt \tag{3}$$

For a viscoelastic isotropic material, tensor $\mathbb{R}$ depends only on two scalar functions $R_\kappa(t)$ and $R_\mu(t)$ which are relaxation functions for compression and shear. The behavior of the material can be expressed by using the following form:

$$\boldsymbol{\sigma}(t) = R_\kappa(t) \circledast tr\dot{\boldsymbol{\epsilon}}(t)\mathbf{1} + 2R_\mu(t) \circledast \dot{\mathbf{e}}(t) \tag{4}$$

where $\mathbf{e}$ is the deviator of the strain tensor.

The viscoelastic constitutive equations of an isotropic viscoelastic material are therefore defined by two relaxation functions: $R_\kappa(t)$ and $R_\mu(t)$.

2.2. Laplace–Carson transform

The Laplace–Carson transform $f^*(p)$ of a real function $f(t),\ t \geqslant 0$ is obtained from its Laplace transform $\tilde{f}(s)$ by:

$$f^*(s) = s\tilde{f}(s) = s\int_0^\infty e^{-st} f(t)dt \tag{5}$$

Effecting the Laplace–Carson transform of the first expression in (4) leads to:

$$\boldsymbol{\sigma}^*(s) = R^*_\kappa(s) tr\boldsymbol{\epsilon}^*(s)\mathbf{1} + 2R^*_\mu(s)\mathbf{e}^*(s) \tag{6}$$

where s is the Laplace variable.

These expressions show that for any fixed value of s, the stress–strain relation in Laplace–Carson space is formally equivalent to the elasticity constitutive equation of an isotropic elastic material. This constitutes the "correspondence principle".

3. Decoupled forms of the overall properties of elastic periodic composites in specific cases

The paper presents different forms of the overall properties under the form of a series whose all terms are decoupled into two parts: the first part depends only on the microstructure and the second part depends only on the elastic properties. Such a decoupling is possible only in specific cases. So, different cases of series comprising decoupled terms are presented: two different forms (strain formulation and stress formulation) in the case of incompressible media and the strain formulation for a specific case of composite containing compressible materials. An example of result obtained by this method is shown and the main results coming from the literature are presented concerning the convergence of the series.

3.1. Basic equations of the problem

Let us consider a periodic composite built on a periodic cell Ω as in Fig. 1 by translation along the three directions of the space.

One denotes by $2a_i$ $(i = 1,2,3)$, the dimension along direction x_i of a basic parallelepipedic cell. Then the displacement field $\mathbf{u} = \mathbf{u}(\mathbf{x})$, the strain field $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}(\mathbf{x})$ and the stress field $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{x})$ induced by a macroscopic strain tensor $\mathbf{E}$ are solutions of:

$$\begin{cases} \boldsymbol{\epsilon}(\mathbf{x}) = \frac{1}{2}\left\{\nabla \otimes \mathbf{u}(\mathbf{x}) + (\nabla \otimes \mathbf{u}(\mathbf{x}))^t\right\} \\ \nabla \cdot \boldsymbol{\sigma}(\mathbf{x}) = \mathbf{0} \\ \boldsymbol{\sigma}(\mathbf{x}) = \mathbb{C}(\mathbf{x}) : \boldsymbol{\epsilon}(\mathbf{x}) \\ \mathbf{u}(\mathbf{x}) = \mathbf{E}.\mathbf{x} + \mathbf{u}_{per}(\mathbf{x}) \end{cases} \tag{7}$$

where the displacement field $\mathbf{u}_{per}$ is Ω-periodic and $\mathbb{C}(\mathbf{x})$ is the elasticity tensor satisfying the periodicity condition:

$$\begin{cases} \mathbb{C}(\mathbf{x}) = \mathbb{C}(\mathbf{x}+\mathbf{d}) \\ \mathbf{d} = \sum_{i=1}^{3} 2n_i a_i \mathbf{e}_i \end{cases} \tag{8}$$

where n_i is an arbitrary integer. Strain and stress tensors are also periodic:

$$\begin{cases} \boldsymbol{\sigma}(\mathbf{x}) = \boldsymbol{\sigma}(\mathbf{x}+\mathbf{d}) \\ \boldsymbol{\epsilon}(\mathbf{x}) = \boldsymbol{\epsilon}(\mathbf{x}+\mathbf{d}) = \mathbf{E} + \boldsymbol{\epsilon}_{per} \end{cases} \tag{9}$$

3.2. Strain and stress fields in Fourier space

Because of the periodicity of the medium, the solution can be developed into Fourier series, as proposed by Iwakuma and Nemat-Nasser (1983) or Moulinec and Suquet (1994).

Let us consider a periodic function $f(\mathbf{x})$ defined on the cell Ω defined by:

$$\Omega = \left\{\mathbf{x}, -a_j \leqslant x_j \leqslant a_j\ (j = 1,2,3)\right\} \tag{10}$$

with the condition of periodicity: $f(\mathbf{x}) = f(\mathbf{x}+\mathbf{d})$

This function can be expanded into Fourier series as follows:

$$f(\mathbf{x}) = \sum_{\xi} \hat{f}(\xi)e^{i\xi\mathbf{x}}, \quad i = \sqrt{-1} \tag{11}$$

with:

$$\hat{f}(\xi) = \frac{1}{V}\int_V f(\mathbf{x})e^{-i\xi\mathbf{x}}dV, \quad \xi_j = \frac{\pi n_j}{a_j}, \quad (\text{no sum on } j) \tag{12}$$

Let us consider the periodic part $\mathbf{u}_{per}$ of the displacement field $\mathbf{u}$ those constant part is assumed null:

$$\mathbf{u}_{per}(\mathbf{x}) = \sum_{\xi}{}' \widehat{\mathbf{u}_{per}}(\xi)e^{i\xi\mathbf{x}} \tag{13}$$

where a prime on Σ indicates that $n = \sqrt{n_k n_k} = 0$ is excluded from the summation. Each Fourier component is given by:

$$\widehat{\mathbf{u}_{per}}(\xi) = \frac{1}{V}\int_V \mathbf{u}_{per}(\mathbf{x})e^{-i\xi\mathbf{x}}d\mathbf{x} \tag{14}$$

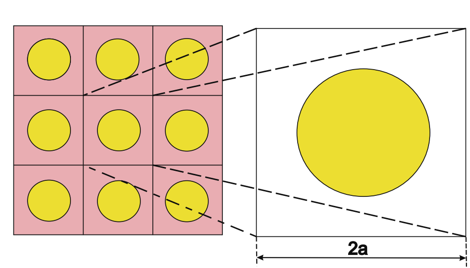

Fig. 1. Basic cell of a heterogeneous periodic medium.

Differentiating this periodic field produces the "periodic strain field" $\boldsymbol{\epsilon}_{per}$ whose name comes from the differentiation of the periodic displacement field. Spatial average of $\boldsymbol{\epsilon}_{per}$ is null and its Fourier expansion is:

$$\boldsymbol{\epsilon}_{per}(\mathbf{x}) = \sum_{\xi}{}' \widehat{\boldsymbol{\epsilon}}_{per}(\xi) e^{i\xi\mathbf{x}} \tag{15}$$

Finally, the relation between these "periodic parts", simultaneously with the constitutive elasticity equation in Fourier space are:

$$\begin{cases} \widehat{\boldsymbol{\epsilon}}_{per}(\xi) = \frac{i}{2}\left\{\xi \otimes \widehat{\mathbf{u}_{per}}(\xi) + \widehat{\mathbf{u}_{per}}(\xi) \otimes \xi\right\} \\ \widehat{\boldsymbol{\sigma}}(\xi) = i\sum_{\zeta} \widehat{\mathbb{C}}(\xi - \zeta) : \left\{\zeta \otimes \widehat{\mathbf{u}}(\zeta)\right\} \end{cases} \tag{16}$$

3.3. Solution of the localization problem by the strain formulation

The solution of the localization problem is obtained by using a "reference medium" whose elasticity tensor is $\mathbb{C}^0$. The derivation of the solution under the form of a Neumann series is given in textbooks (Milton, 2002).

This derivation is recalled in Appendix C for completeness and produces the solution of the cell problem under the form of a Neumann series which can be expressed as:

$$\boldsymbol{\epsilon}(\mathbf{x}) = \{\mathbb{I} - \Gamma * \delta\mathbb{C} + \Gamma * [\delta\mathbb{C}(\Gamma * \delta\mathbb{C})] - \cdots\} : \mathbf{E} \tag{17}$$

where Γ is the "strain Green's tensor" whose expression in the case of an isotropic reference material is recalled in Appendix A.

In this expression, $\delta\mathbb{C}$ is given by:

$$\delta\mathbb{C}(\mathbf{x}) = \left(\mathbb{C}_I - \mathbb{C}^0\right) I_I(\mathbf{x}) + \left(\mathbb{C}_M - \mathbb{C}^0\right) I_M(\mathbf{x}) \tag{18}$$

In the following, the elastic properties of the "reference medium" are taken as those of the matrix.

3.4. A decoupled formulation of the local strain in the case of incompressible constituents

In the case of isotropic incompressible constituent materials and using the previously defined reference medium, $\delta\mathbb{C}$ must be replaced by the part of the elasticity tensor which concerns only deviatoric parts of stresses and strains, given by:

$$\delta\mathbb{C}(\mathbf{x}) = -2\delta_{\mu} I_I(\mathbf{x})\mathbb{I} \tag{19}$$

where $\delta_{\mu} = \mu_M - \mu_I$ and $\mathbb{I}$ is the fourth order identity tensor.

Moreover, in this case the Green's tensor is a transversely isotropic tensor which can be expressed, as recalled in Appendix A, as:

$$\boldsymbol{\Gamma}(\mathbf{x}) = \frac{\mathbb{E}_4}{2\mu_M} \tag{20}$$

where $\mathbb{E}_4$ is a function of wave vectors given in Appendix A.

Substituting (19), (20) into (17) leads to:

$$\boldsymbol{\epsilon}(\mathbf{x}) = \left\{\mathbb{I} + \frac{\delta_{\mu}}{\mu_M}\mathbb{E}_4 * I_I + \left(\frac{\delta_{\mu}}{\mu_M}\right)^2 \mathbb{E}_4 * [I_I(\mathbb{E}_4 * I_I)] - \cdots\right\} : \mathbf{E} \tag{21}$$

In this relation, only the deviatoric parts of the strains are significant, because the volumetric strains are null. Under this form, it can be seen that all terms are decoupled, being the products of one part which is only a function of the distribution of the heterogeneities, through function I_I, and another part which is only a function of the shear moduli. This feature will be conserved in the following and will lead to the solution for effective viscoelastic properties. It is worthwhile mentioning that a weaker version of such a decoupling was mentioned previously by Iwakuma and Nemat-Nasser (1983) in the case of the most general isotropic material. However, such a weaker coupling does not allow to provide in the most general case the same kind of solution as the one given in the following section for viscoelastic media. Such a full decoupling is the key of the viscoelastic solution, as it will be seen thereafter. In the following subsection, the homogenized elasticity tensor will be provided in the case of incompressible constituents.

3.5. Homogenization with the strain formulation for incompressible materials

The behavior at the macroscopic scale can be written from deviatoric parts of macroscopic stress tensor at convergence, which will be denoted thereafter by $\boldsymbol{\Sigma}$. It is given as:

$$\boldsymbol{\Sigma} = \left\langle (2\mu_I \cdot I_I + 2\mu_M \cdot I_M)\boldsymbol{\epsilon}(\mathbf{x})\right\rangle_V \tag{22}$$

In (22), the notation $\langle\Phi\rangle$ is used for referring to the volume average of Φ:

$$\langle\Phi\rangle_V = \frac{1}{V}\int_V \Phi dx \tag{23}$$

The macroscopic deviatoric stress in (22) can be split into partial deviatoric stresses:

$$\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_I + \boldsymbol{\Sigma}_M \tag{24}$$

with:

$$\begin{cases} \boldsymbol{\Sigma}_I = 2\mu_I \langle I_I(\mathbf{x}) \cdot \boldsymbol{\epsilon}^{\infty}(\mathbf{x})\rangle_I \\ \boldsymbol{\Sigma}_M = 2\mu_M : \langle I_M(\mathbf{x}) \cdot \boldsymbol{\epsilon}^{\infty}(\mathbf{x})\rangle_M \end{cases} \tag{25}$$

We set:

$$\begin{cases} \mathbb{A}_0 = f_I \mathbb{I} \\ \mathbb{B}_0 = f_M \mathbb{I} \\ \mathbb{A}_i = \left\langle I_I \cdot \left\{\mathbb{E}_4 * \left[I_I \ldots \underbrace{(\mathbb{E}_4 * I_I)}_{i}\right]\right\}\right\rangle_I \\ \mathbb{B}_i = \left\langle I_M \cdot \left\{\mathbb{E}_4 * \left[I_I \ldots \underbrace{(\mathbb{E}_4 * I_I)}_{i}\right]\right\}\right\rangle_M \end{cases} \tag{26}$$

Then the effective tensor is given by:

$$\mathbb{C}^{eff} = 2\mu_I\left(\mathbb{A}_0 + \sum_i \left(\frac{\delta_{\mu}}{\mu_M}\right)^i \mathbb{A}_i\right) + 2\mu_M\left(\mathbb{B}_0 + \sum_i \left(\frac{\delta_{\mu}}{\mu_M}\right)^i \mathbb{B}_i\right) \tag{27}$$

Note that, if we define $\mathbb{F}_i(\mathbf{x})$ by:

$$\mathbb{F}_i(\mathbf{x}) = \mathbb{E}_4 * \left[I_I \ldots \underbrace{(\mathbb{E}_4 * I_I)}_{i}\right] \tag{28}$$

Then:

$$\mathbb{A}_i + \mathbb{B}_i = \langle\mathbb{F}_i(\mathbf{x})\rangle_V \tag{29}$$

From another point of view, $\langle\mathbb{F}_i(\mathbf{x})\rangle_V$ is given by the value of the Fourier transform at point $\xi = 0$ as:

$$\langle\mathbb{F}_i(\mathbf{x})\rangle = \hat{\mathbb{F}}_i\ (\xi = 0) \tag{30}$$

with:

$$\hat{\mathbb{F}}_i(\xi) = \mathbb{E}_4\left[I_I * \ldots * \underbrace{(\mathbb{E}_4 \cdot I_I)}_{i}\right] \tag{31}$$

Tensor $\mathbb{E}_4$ being null for a null wavenumber, it leads to:

$$\mathbb{A}_i + \mathbb{B}_i = 0 \tag{32}$$

The effective elasticity tensor is finally given by:

$$\mathbb{C}^{eff} = 2\mu_M\mathbb{I} + 2\sum_{i=0}^{\infty}\frac{(\delta_{\mu})^{i+1}}{(\mu_M)^i}\mathbb{A}_i \tag{33}$$

Under this form, the effective elasticity tensor appears again as composed of terms which are decoupled into distribution dependent terms and shear moduli dependent terms.. Indeed, the determination of tensors $\mathbb{A}_i$ depends only on the geometry of the basic cell through tensor I_I. This form allows to obtain in the following section the effective viscoelastic properties of the composite. Before to proceed with this step, it is worthwhile mentioning that the strain formulation described previously is conditionally convergent, as it will be seen in Section 3.9. More particularly, when the inclusion is largely stiffer than the matrix, the use of the matrix properties for defining the reference material is no more possible, as it will be shown thereafter.

In this case, a first solution may be to use the inclusion modulus as reference modulus. Indeed, a formula similar to Eq. (21) can be obtained by noticing that $I_M(\xi) + I_I(\xi) = 0$ for any non-null wavevector. This new series solution does not converge for the same kind of contrast between inclusion and matrix properties, as seen thereafter.

Alternatively, a stress formulation can be used. The following section shows that this stress formulation can also lead to the effective elasticity tensor under the form of a Neumann series with decoupled terms.

3.6. The series solution with decoupling using the stress formulation

The stress formulation is based on a Green's tensor which is different from the one used in the strain formulation. This new Green's tensor $\mathbf{\Lambda}$ is given for an incompressible material by:

$$\mathbf{\Lambda} = 2\mu_0(3\mathbb{E}_1 + \mathbb{E}_3) \tag{34}$$

where $\mathbb{E}_1$ and $\mathbb{E}_3$ are again elements of the Walpole's basis defined in Appendix A.

The following steps are similar to the ones defined previously by using again the properties of the matrix as the reference medium and the stress Green's tensor. The iterative scheme of the stress approach in this case is given by:

$$\boldsymbol{\sigma}^{i+1}(\mathbf{x}) = \mathbf{\Sigma} - \mathbf{\Lambda}(\mathbf{x}) * \left(\delta\mathbb{S} : \boldsymbol{\sigma}^i(\mathbf{x})\right) \tag{35}$$

where tensor $\mathbb{S}$ is the compliance tensor. In order to facilitate the construction of the Neumann series with decoupled terms, the matrix is selected again for the reference medium. Thus, the iterative scheme can be reduced to:

$$\boldsymbol{\sigma}^{i+1}(\mathbf{x}) = \mathbf{\Sigma} - \frac{\delta_\mu}{\mu_I}\mathbb{E}' * (I_I \cdot \boldsymbol{\sigma}^i(\mathbf{x})) \tag{36}$$

where $\mathbb{E}' = 3\mathbb{E}_1 + \mathbb{E}_3$. The expression for the effective compliance related to deviatoric parts of stresses and strains is then given by:

$$\mathbb{S}^{eff} = \frac{1}{2\mu_M}\left(\mathbb{I} + \sum_{i=0}^{\infty}(-1)^i\left(\frac{\delta_\mu}{\mu_I}\right)^{i+1}\mathbb{A}'_i\right) \tag{37}$$

where tensor $\mathbb{A}'_i$ is defined by:

$$\mathbb{A}'_i = \begin{cases} f_I\mathbb{I} & (i=0) \\ \left\langle I_I \cdot \left\{\mathbb{E}' * \left[I_I \ldots \underbrace{(\mathbb{E}' * I_I)}_{i}\right]\right\}\right\rangle_I & (i>0) \end{cases} \tag{38}$$

This expression of the effective compliance related to deviatoric stresses and strains is again obtained under the form of a series with decoupled terms. The method which is described in the next section for obtaining the effective properties of viscoelastic media can be readily extended by using this stress formulation. As it will be seen thereafter, the convergence domains of these two series are not the same and the use of the stress formulation can be needed for very stiff inclusions.

3.7. Series solution with decoupling for specific compressible materials

The previous subsections have provided the effective tensors of elastic heterogeneous materials for incompressible materials. However, the case of composite made of arbitrary compressible materials cannot be formulated under a similar form because in this case, the elasticity problem must be treated with the full Green's tensor which comprises two terms related to different elastic moduli, as seen in Appendix A. Therefore, it is generally no more possible to separate the contribution of the elastic moduli from the one of the geometry. However, a case which can lead to a similar solution is the one where both phases have the same Poisson's ratio.

In this previously defined specific case, the Green's tensor is written again by taking the elastic tensor of the matrix to define the reference medium as:

$$\mathbf{\Gamma}_0 = \frac{1}{\lambda_M + 2\mu_M}\mathbb{E}_2 + \frac{1}{2\mu_M}\mathbb{E}_4 = \frac{1}{2\mu_M}\mathbf{\Gamma}' \tag{39}$$

with:

$$\mathbf{\Gamma}\prime = c(v)\mathbb{E}_2 + \mathbb{E}_4 \tag{40}$$

$$c(v) = \frac{1-2v}{1-v} \tag{41}$$

The elastic tensors of both phases can be written as:

$$\mathbb{C} = 3\kappa\mathbb{J} + 2\mu\mathbb{K} = 2\mu\mathbb{C}' \tag{42}$$

where:

$$\mathbb{C}' = d(v)\mathbb{J} + \mathbb{I} \tag{43}$$

$$d(v) = \frac{1+v}{1-2v} \tag{44}$$

Then, the iterative scheme is:

$$\boldsymbol{\epsilon}^{(i+1)}(\mathbf{x}) = \mathbb{E} - \frac{\mu_I - \mu_M}{\mu_M}\mathbf{\Gamma}' * \left(I_I\mathbb{C}'\boldsymbol{\epsilon}^{(i)}(\mathbf{x})\right) \tag{45}$$

This form is similar to the one obtained for incompressible materials, the only difference being in the computation of the tensorial coefficients $\mathbb{A}_i$ which now depend not only on the geometry, but also on the Poisson's ratio through tensor $\mathbf{\Gamma}'$.

Obviously, this configuration is not physically realistic, because when using it in the following section for the extension to viscoelastic cases, it must be noticed that v must be the same for inclusion and matrix, which means that the Poisson's ratio for any value of the "equivalent elastic moduli" obtained by the correspondence principle is constant when Laplace variable s varies. However, it can constitute a convenient way for checking fully numerical alternative solutions.

3.8. Numerical example in the case of elasticity

Obtaining the viscoelastic properties of the composite rests on the solution obtained in elasticity. Before studying the extension to viscoelasticity, results on an example of elastic composite will be presented.

The case under study is related to the 2D problem of the elasticity components of a composite made of fibers distributed along a simple squared lattice (Fig. 5. The materials are assumed isotropic, linearly elastic and incompressible. The shear moduli of the components are: $\mu_M = 70$ GPa and $\mu_I = 5$ GPa.

The shape functions I_I in the calculation of $\mathbb{A}_m$ are given by (Nemat-Nasser et al., 1982):

$$I_I(\xi) = \frac{2 \cdot S_I J_1(\eta)}{\eta} \tag{46}$$

where J_1 is the first order Bessel function. S_I is the surface of the inclusion, η is given by:

$$\eta = R\left(\xi_1^2 + \xi_2^2\right)^{1/2} \tag{47}$$

where R is the radius of the fiber.

As an example, Fig. 2 shows the values of C^{eff}_{2323} as a function of the concentration of fibers compared to Voigt and Reuss bounds and to the simplified solution proposed in Nemat-Nasser et al. (1982). As the simplified solution will be used in the following, it is worthwhile mentioning that the complete solution is close to the simplified solution for lower concentrations, but departs from this simplified solution at higher concentrations. Having studied in previous works this comparison in the case of elasticity for numerous kinds of heterogeneous media, it is possible to know when the simplified solution leads to an exact value of elasticity tensor. These results will be used in the following.

From a numerical point of view, the solution of the Neumann series depends mainly on the number of terms N_k in the approximation retained for the Fourier transform and on the number of terms N_s retained in the Neumann series. Satisfying results were obtained in the case of Fig. 2 by using $N_k = 64 \times 64$. This value will be kept in the sequel where only the relation between the physical or geometrical parameters and N_s will be studied.

3.9. Results on convergence obtained from the literature

Classical results are already known on the convergence of the Neumann series. Restricting to the case of incompressible materials, these results can be summarized as follows:

- The Neumann series with the basic scheme (strain formulation) is conditionally convergent. Based on a study of the spectrum of the iteration operator, the convergence is ensured (sufficient condition) if the shear modulus of the reference medium μ_0 complies with (Michel et al., 2001):

$$\mu_0 > \frac{\mu(\mathbf{x})}{2} > 0 \tag{48}$$

for any point $\mathbf{x}$ within the cell. For obtaining the decoupling in an inclusion-matrix composite, this condition leads to:

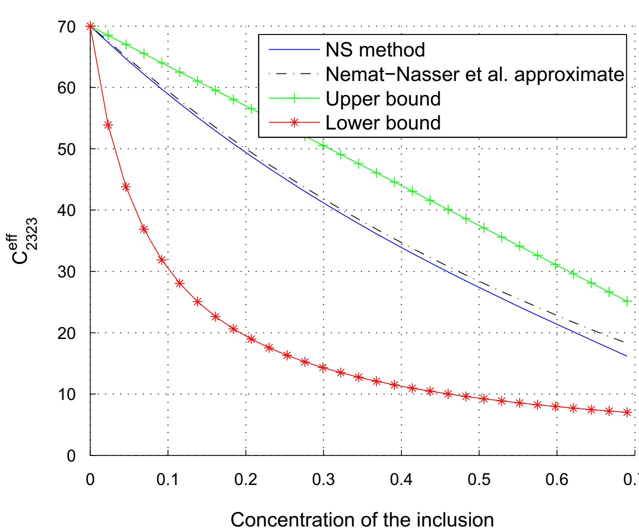

Fig. 2. Effective shear modulus obtained from the NS method compared with the solution obtained by Iwakuma and Nemat-Nasser approximate. Both are compared with Voigt and Reuss bounds.

$$2\mu_M > \mu_I > 0 \tag{49}$$

for the material of the matrix being used as a reference medium or

$$\infty > 2\mu_I > \mu_M > 0 \tag{50}$$

for the material of the inclusions being chosen as a reference medium.

- The convergence of the Neumann series for the dual scheme (stress formulation) is also conditionally convergent. The convergence is ensured if the shear modulus of the reference medium complies with:

$$\frac{1}{\mu_0} > \frac{1}{2\mu(\mathbf{x})} > 0 \tag{51}$$

- The previous conditions exclude the cases when the contrast is infinite, i.e. for μ_I infinite or null. For the second case, there are some divergences between authors, but some results have shown that the convergence of the effective properties (maybe without a strict convergence of the strain tensor at any point of the cell) can be obtained for some geometries even if the contrast is infinite ($\mu_I = 0$). This point will be studied thereafter.
- When the convergence is ensured for a finite contrast, the terms of the series decrease as those of a geometrical series (Milton, 2002). The convergence criteria is related to the eigenvalue of the iteration operator having the highest absolute value.
- The strain tensor induced by the basic scheme is always compatible. As a consequence, a classical way to ensure the convergence of the local strain field is to ensure that the residue on the equilibrium equation is small. Alternatively, the convergence of the stress iterative scheme must be checked against the compatibility of the strains obtained at each iteration. Generally, the convergence of effective properties is obtained largely before the convergence of the local equilibrium or compatibility.

3.10. Results on convergence obtained from the decoupled formulation

A first result which can be recovered is the main feature of Neumann's series which is that the rate of convergence is the one of a geometrical series. This result is readily recovered from the expression (44) of the Green's tensor. If one considers that tensors $\mathbb{A}_i$ are bounded by $\mathbb{A}_M$, the generic term of the series is bounded by $2\mathbb{A}_M \frac{(\delta_\mu)^{i+1}}{(\mu_M)^i}$ which is clearly a geometrical series.

The convergence of the series as the one of a geometrical series is classical, but the usual formulation does not allow to know the contribution of the geometry of the heterogeneities in the different terms of the series. With the decoupled expression of the terms of the series, the contribution of the geometry can be studied by looking at the components of the differents terms $\mathbb{A}_i$ appearing in the series, because these terms depend only on the geometry of the heterogeneities. As an example, Fig. 3 displays the values of terms A_{1212} in a semi-logarithmic plot. It can be seen that these terms are aligned along a straight line for a sufficient number of iterations, which means that they behave as a geometrical series. This result explains that for such a geometry, the series is convergent even if the shear modulus of the inclusions is null. Indeed, in this case, the contributions to the current term of the series of the part related to the shear moduli is equal to 1, which does not ensure the convergence only under the classical assumption of bounded $\mathbb{A}_i$. It is clearly the decay of the $\mathbb{A}_i$ terms as those of a geometrical series which can ensure the convergence.

As a consequence, for every geometry such as the components of $\mathbb{A}_i$ converge as a geometrical series, the Neumann series converges even for a null value of the shear modulus of the inclusions. It is worthwhile noticing that such a property may be true only for some specific geometries of the distributions of the components

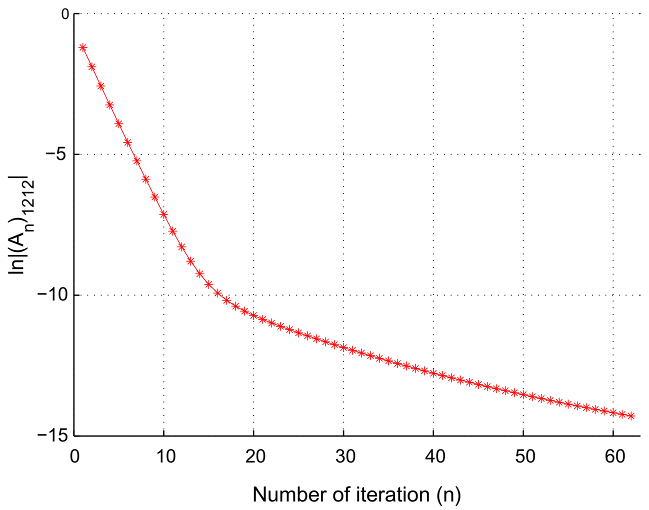

Fig. 3. Logarithm of the geometry-dependent terms A_n for elastic coefficient C_{1212} as a function of the number of iterations.

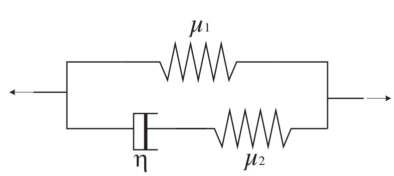

Fig. 4. Viscoelastic rheological scheme of Zener's type.

and that in the most general case, without a specific study, the condition excluding the null value of shear modulus must be used.

4. Series solution for the viscoelastic properties

As seen before, the effective shear moduli related to the homogenization problem can be expressed as a series of terms decoupling the moduli of the phases and the geometry of the inclusions. In addition, all components containing the moduli of the phases are polynomial fractions of the Laplace variable. If the viscoelastic shear moduli of the phases are themselves expressed as polynomial fractions of the Laplace variable, as for usual viscoelastic materials, it means that all terms of the series are polynomial fractions, whose inverse Laplace transform can be computed explicitly. In the first part of this section, we consider therefore an example of such a viscoelastic incompressible material made of a viscoelastic matrix of Zener type containing elastic inclusions. In the second part of this section, the convergence conditions are studied.

4.1. The explicit relaxation kernel using the strain formulation

Let us consider an incompressible composite of inclusion-matrix type where the matrix is viscoelastic of Zener type and contains elastic inclusions. The behavior of the matrix in the Laplace–Carson's space is described by the relationship between the deviators of stresses and strains in the form:

$$\mathbf{s} = 2\mu(s)\boldsymbol{e} \tag{52}$$

where coefficient $\mu(s)$ is the shear modulus in Laplace–Carson space defined by:

$$\mu_M = \mu(s) = \frac{\mu_1(\mu_2 + \eta s) + \mu_2 \eta s}{\mu_2 + \eta s} \tag{53}$$

Using the matrix as a reference medium and the strain formulation, the effective tensor in Laplace–Carson's space is given from (33) by:

$$\mathbb{C}^{eff}(s) = 2\mu(s)\left[\mathbb{I} + \sum_{i=1}^{\infty}\left(\frac{\mu(s) - \mu_I}{\mu(s)}\right)^i \mathbb{A}_i\right] \tag{54}$$

where tensors $\mathbb{A}_i$ can be determined independently by (26) and depend only of the geometry of the periodic cell.

The objective is now to obtain explicitly the inverse Laplace transforms of all terms of the series. We note that for a Zener-type material, the shear modulus of the matrix in the Laplace–Carson space has the form:

$$\mu(s) = \frac{A + Bs}{C + Ds} \tag{55}$$

where A, B, C, D depend on the elastic and viscous moduli. We put:

$$s^* = \frac{Bs}{A} \tag{56}$$

which gives:

$$\mu(s) = \frac{B}{D}\left(\frac{s^* + 1}{s^* + b}\right) \tag{57}$$

$$\mu(s) - \mu_I = \frac{EB + FAs^*}{CB + DAs^*} \tag{58}$$

and:

$$\frac{\mu(s) - \mu_I}{\mu(s)} = \frac{BE + FAs^*}{AB(s^* + 1)} \tag{59}$$

with:

$$\begin{cases} E = A - \mu_I C \\ F = B - \mu_I D \\ b = \frac{CB}{DA} \end{cases} \tag{60}$$

The Laplace transform of the effective modulus $\hat{\mathbb{C}}^{eff}(s)$, is defined by:

$$\hat{\mathbb{C}}^{eff}(s) = \frac{\mathbb{C}^{eff}(s)}{s} \tag{61}$$

The relaxation function in the Laplace space becomes:

$$\hat{\mathbb{C}}^{eff}(s^*) = \frac{B\mu_I\mathbb{A}_0}{As^*} + \frac{B^2(2\mathbb{I} - \mathbb{A}_0)}{AD}\,\frac{s^* + 1}{s^*(s^* + b)} - \sum_{i=1}^{m}\frac{\mathbb{A}_i B\alpha^i\beta}{A} \times \frac{(s^* + a)^{i+1}}{(s^* + 1)^i(s^* + b)s^*} \tag{62}$$

where:

$$\alpha = \frac{F}{B} \quad \beta = \frac{F}{D} \quad a = \frac{BE}{AF} \tag{63}$$

which gives:

$$\hat{\mathbb{C}}^{eff}(s^*) = \mathbb{G}_1\frac{1}{s^*} + \mathbb{G}_2\frac{1}{s^* + b} - \sum_{i=1}^{m}\mathbb{K}_i\hat{L}^{(i)} \tag{64}$$

with:

$$\begin{aligned} \mathbb{G}_1 &= \frac{B\mu_I}{A}\mathbb{A}_0 + \frac{B^2}{bAD}(2\mathbb{I} - \mathbb{A}_0) \\ \mathbb{G}_2 &= \frac{B^2(b-1)}{ADb}(2\mathbb{I} - \mathbb{A}_0) \\ \mathbb{K}_i &= \frac{B\alpha^i\beta}{A}\mathbb{A}_i \\ \hat{L}^{(i)} &= \frac{(s^* + a)^{i+1}}{(s^* + 1)^i(s^* + b)s^*} \end{aligned} \tag{65}$$

The polynomial fraction $\hat{L}^{(i)}$ can be split into partial terms as:

$$\hat{L}^{(i)} = \frac{H_1^{(i)}}{s^*} + \frac{H_2^{(i)}}{s^* + b} + \sum_{j=1}^{i} \frac{M_j(i)}{(s^* + 1)^j} \tag{66}$$

where the coefficients $H_1,\ H_2,\ M_j(i)$ are determined from the poles: $s^* = 0,\ s^* = -b,\ s^* = -1$:

– For the pole $s^* = 0$:

$$H_1^{(i)} = \lim_{s^* \to 0} \left\{ s^* \widehat{L}^{(i)} \right\} = \frac{a^{i+1}}{b} \tag{67}$$

– For the pole $s^* = -b$:

$$H_2^{(i)} = \lim_{s^* \to -b} \left\{ \frac{1}{s^*} \left(\frac{s^* + a}{s^* + 1} \right)^i (s^* + a) \right\} = \frac{(a-b)^{i+1}}{-b(1-b)^i} \tag{68}$$

– For the pole $s^* = -1$:

$$M_j(i) = \lim_{s^* \to -1} \left\{ \frac{1}{(i-j)!} \frac{d^{(i-j)}}{(ds^*)^{(i-j)}} \left[\frac{(s^* + a)^{i+1}}{s^*(s^* + b)} \right] \right\} \tag{69}$$

which gives:

$$M_j(i) = \frac{1}{b} \frac{1}{(i-j)!} \left(M_j^1 + M_j^2 \right) \tag{70}$$

with:

$$M_j^1(i) = \lim_{s^* \to -1} \left\{ \frac{d^{(i-j)}}{(ds^*)^{(i-j)}} \left[\frac{(s^* + a)^{i+1}}{s^*} \right] \right\} \tag{71}$$

$$M_j^2(i) = -\lim_{s^* \to -1} \left\{ \frac{d^{(i-j)}}{(ds^*)^{(i-j)}} \left[\frac{(s^* + a)^{i+1}}{s^* + b} \right] \right\} \tag{72}$$

Using Leibniz's formula leads to:

$$M_j^1(i) = -\sum_{r=0}^{i-j} C_r^{i+1} (i-j)! (a-1)^{i-r+1} \tag{73}$$

$$M_j^2(i) = \sum_{r=0}^{i-j} C_r^{i+1} (i-j)! (a-1)^{i-r+1} (-1)^{i-j-r} \left(\frac{1}{b-1} \right)^{i-j-r+1} \tag{74}$$

Therefore:

$$M_j(i) = -\frac{1}{b} \sum_{r=0}^{i-j} C_r^{i+1} (a-1)^{i-r+1} \left[1 + \left(\frac{-1}{b-1} \right)^{i-j-r+1} \right] \tag{75}$$

where C_n^k is the binomial coefficient defined as: $C_n^k = \frac{n!}{k!(n-k)!}$.

The relaxation function is obtained by combining the different terms appearing in the Laplace transform:

$$\hat{\mathbb{C}}^{eff}(s^*) = \mathbb{R}_1 \frac{1}{s^*} + \mathbb{R}_2 \frac{1}{s^* + b} - \sum_{i=1}^{m} \sum_{j=1}^{i} \mathbb{K}_i M_j(i) \frac{1}{(s^* + 1)^j} \tag{76}$$

with:

$$\mathbb{R}_1 = \mathbb{G}_1 - \sum_{i=1}^{m} \mathbb{K}_i H_1^{(i)} \tag{77}$$

$$\mathbb{R}_2 = \mathbb{G}_2 - \sum_{i=1}^{m} \mathbb{K}_i H_2^{(i)} \tag{78}$$

The inverse Laplace transform is performed by using:

$$L^{-1}\left\{ F\left(\frac{s}{k} \right) \right\} = k f(kt) \tag{79}$$

which leads to the relaxation function in non-dimensional time t^* which is determined explicitly by the following formula:

$$\mathbb{C}^{eff}(t^*) = \mathbb{R}_1 + \mathbb{R}_2 e^{-bt^*} - e^{-t^*} \sum_{i=1}^{m} \sum_{j=1}^{i} \mathbb{K}_i M_j \frac{(t^*)^{j-1}}{(j-1)!} \tag{80}$$

Finally, the relaxation function is given explicitly by:

$$\mathbb{C}^{eff}(t) = \frac{A}{B} \left\{ \mathbb{R}_1 + \mathbb{R}_2 e^{-\frac{bAt}{B}} - e^{-\frac{At}{B}} \sum_{i=1}^{m} \sum_{j=1}^{i} \mathbb{K}_i M_j \frac{\left(\frac{At}{B}\right)^{j-1}}{(j-1)!} \right\} \tag{81}$$

This expression of the effective relaxation function of the composite material is given explicitly as a function of time.

An interesting question is the mathematical nature of this final solution. A model which is often used for modeling inclusion-matrix composites is the generalized self-consistent scheme. It was shown in Beurthey et al. (2000) that the viscoelastic behavior obtained from this model is characterized not only by discrete relaxation times, but also by a continuous bounded relaxation spectrum. In comparison, the partial sums obtained in the present paper are characterized only by two discrete relaxation times as it can be seen in Eqs. (77) and (82). However, the second relaxation time is related to a multiple pole as it can be seen in (77). The multiplicity of the poles becomes very large when the number of terms of the series tends to infinity. The mathematical structure of the present solution seems to be quite different from the one obtained by using the generalized self-consistent scheme.

4.1.1. Convergence conditions

The form of the series providing the relaxation kernel in time domain is more complex than the form of the effective elasticity tensor in the Laplace space. The elasticity tensor in Laplace space is similar to the one obtained in the case of elasticity, but with moduli which are a priori complex, because the result of Laplace transform is usually defined in the complex plane. A necessary condition for the convergence of the elasticity tensor in Laplace space is that the convergence is ensured for any positive real value of the Laplace variable s, i.e. for any value of the "equivalent elasticity modulus" in Laplace space for positive real values of s.

From another point of view, the inversion of the Laplace transform can be produced from the "restricted" Laplace transform produced by using only the positive real values of Laplace variables. This was used recently by Indratno and Ramm (2009) which produced a numerical inversion of Laplace transform using only real positive values of Laplace variable. Indeed, the complex values of Laplace variables are used mainly for commodity, through other practical inversion tools (Bromwich contour, Padé's approximant,...), but the use of all the complex plane for inversion is not strictly necessary.

The previous necessary condition becomes a sufficient condition for constructing the Laplace transform on the real axis and we will assume that the construction of such a "restricted" Laplace transform of the series is sufficient for the convergence of this series in time domain. Using for example the modulus of the matrix as reference modulus and the Zener model, the previous condition becomes:

$$2\mu(s) > \mu_l > 0 \tag{82}$$

For s real, the equivalent modulus $\mu(s)$ is a monotonous function of s and the convergence condition must be met for $s = 0$ and $s = \infty$. Accepting a null value of μ_l for the reasons explained in the previous section, this leads to:

$$0 \leqslant \mu_l < 2\mu_1 \tag{83}$$

and

$$0 \leqslant \mu_I < 2(\mu_1 + \mu_2) \tag{84}$$

and finally the first of these conditions is the most constraining.

Contrarily to the conditions obtained in the case of elasticity, it was not completely proved that this condition, despite being based on reasonable assumptions, is a sufficient condition of convergence. However, it was found that this condition leads to the convergence in time domain for all tests which were performed during the present study.

Obviously, similar conditions can be obtained for the stress formulation or when the modulus of the inclusion is chosen as reference.

It is worthwhile recalling that the possibility of having $\mu_I = 0$ depends on the convergence of A_i terms and may not be possible for any geometries of heterogeneities.

5. Numerical application

5.1. Fiber composite

In this 2*D* numerical example, the macroscopic behavior of a viscoelastic matrix of Zener type containing fibers whose cross sections are circular is studied as previously in the elastic case. The sections of the fibers are distributed along a squared lattice (Fig. 5). The physical parameters are defined in Table 1. The number of wave vectors used in the numerical tests is 64×64 for all cases under study. The number of terms retained in the series is determined for obtaining an accuracy of 10^{-4}. For ensuring the convergence, the cases of lowest and highest moduli induced by Zener's expression of the shear modulus on the real axis are taken into account:

$$N = Sup\{N(Sup\{\mu_M(s)\}), N(Inf\{\mu_M(s)\})\} \tag{85}$$

The 3D effective tensor of the overall material, taking into account the incompressibility, comprises 3 coefficients. The time dependence of two of these components is given in Figs. 6 and 7. The results obtained by using the simplified method described in Hoang-Duc and Bonnet (2012) are also reported. As explained in the previous section, the approximate FFT method described in Iwakuma and Nemat-Nasser (1983) leads to the same results as the complete FFT method for a large range of elasticity tensors. The approximate FFT method was extended to viscoelasticity in Hoang-Duc and Bonnet (2012). The simplified method was proved to produce satisfying results in elasticity for moderate values of concentrations and contrasts (let us say for $C_1 < \mu_M/\mu_I < C_2$). Using again the restriction of the Laplace transform to the real axis, both methods produce therefore the same "restricted" Laplace transforms as soon as $C_1 < \mu_M(s)/\mu_I < C_2$. Under this assumption, the simplified FFT method can be considered as producing reference results.

Thus, the results produced by using the Neumann series are compared with those obtained by the simplified FFT method described in Hoang-Duc and Bonnet (2012). The contrast between the phases and the concentration of those phases were obviously chosen in order to be in the range of validity of the simplified FFT method. The results are the same for both cases. Taking into account the strong differences between both formulations, it can be concluded that the new method produces fine results.

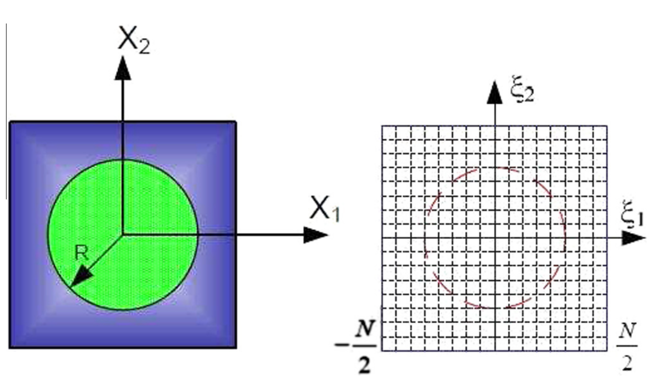

Fig. 5. Basic periodic cell of the fiber composite with circular cross sections.

Table 1
Mechanical properties of an incompressible composite with a viscoelastic matrix of Zener type.

Phase	Inclu.	Matrix	
		Ele. 1	Ele. 2
Concentration (%)	30	70	
Shear modulus (GPa)	10	70	50
Shear viscosity (GPa day)	0	0.2	

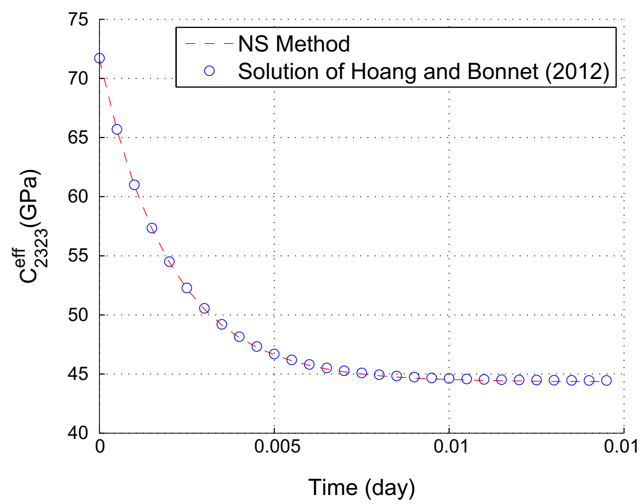

Fig. 6. Component $C^{eff}_{2323}(t)$ as a function of time, determined by the NS method and by the FFT approximate solution of Hoang-Duc and Bonnet (2012).

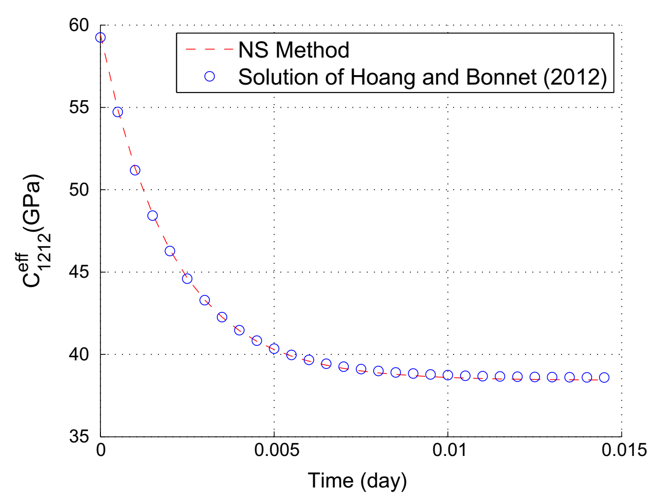

Fig. 7. Component $C^{eff}_{1212}(t)$ as a function of time, determined by the NS method and by the FFT approximate solution of Hoang-Duc and Bonnet (2012).

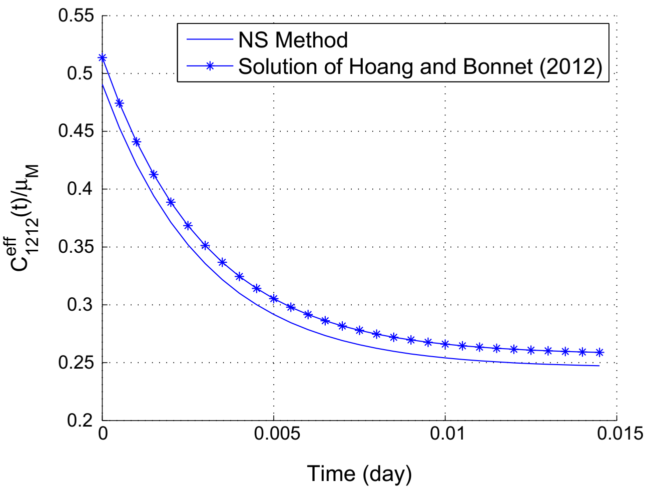

Fig. 8. Comparison of NS solution and approximate solution for a matrix containing inclusions made of an incompressible fluid.

5.2. Saturated porous medium

As explained in the previous section, the convergence of the Neumann series is ensured also for the case of a null value of the inclusions modulus, at least for the geometry of microstructure used in the example.

So, an extreme case of a fiber composite has been treated, where an infinite contrast is obtained by filling the inclusions with an incompressible fluid having a null shear modulus.The viscoelastic behavior of the matrix is again of Zener type (Fig. 4). The relaxation time is given by $\tau = \frac{\eta}{\mu_M} = 0.003$ days.

Fig. 8 displays the evolution of the ratio $\frac{C^{eff}_{1212}(t)}{\mu_M}$ with time. For this new result, it may be seen that the result obtained by the complete solution departs from the one coming from the simplified solution, as waited from the results obtained in elasticity.

The numerical results which were obtained for the simple periodic microstructures under study are characterized by a fully anisotropic behavior. In comparison, the results coming for example from the generalized self-consistent theory (Beurthey et al., 2000) lead to isotropic effective properties for isotropic properties of the constituents. The difference comes obviously from the fact that both models do not correspond to the same microstructure. The generalized self-consistent theory approximates the interaction between one given inclusion and the other ones, while the method proposed in the present paper takes fully into account the interactions between all inclusions contained in the periodic medium.

6. Conclusion

A new solution for obtaining the effective properties of heterogeneous incompressible viscoelastic media made of isotropic viscoelastic constituents has been presented. This solution is based on a classical formulation provided in the case of elasticity, which uses the Green's tensor in Fourier domain and the Neumann series. The series for the elastic solution has been rewritten in a form where all terms are decoupled: they comprise a geometrically dependent part and a part depending on the physical properties. A similar form is obtained for the viscoelastic composite in Laplace–Carson domain. For usual rheological viscoelastic models, each term can then be written as a polynomial fraction of the Laplace variable, leading to an explicit expression of the inverse Laplace transform. A comparison leading to satisfying results has been made with a previous solution. The solution presents itself under different forms, depending on the formulation used (stress formulation or strain formulation) and also depending on the value of reference medium used for the solution (matrix or inclusion). A solution has been also given for compressible component phases under restrictive conditions of the properties of both phases.

Appendix A. Green's tensors for an isotropic reference elastic medium

The strain Green's tensor for an isotropic reference elastic medium having Lamé constants λ and μ is given by:

$$\mathbf{\Gamma} = \frac{1}{\lambda + 2\mu}\mathbb{E}_2 + \frac{1}{2\mu}\mathbb{E}_4 \tag{A.1}$$

where tensors $\mathbb{E}_2$ and $\mathbb{E}_4$ are defined below as components of the Walpole's basis.

In the case of an incompressible material, this tensor reduces to:

$$\mathbf{\Gamma} = \frac{1}{2\mu}(\mathbb{E}_2 + \mathbb{E}_4) \tag{A.2}$$

The stress Green's tensor is similarly given by:

$$\mathbf{\Lambda} = \frac{2\mu(3\lambda + 2\mu)}{\lambda + 2\mu}\mathbb{E}_1 + 2\mu\mathbb{E}_3 \tag{A.3}$$

In the case of an incompressible material, this tensor reduces to:

$$\mathbf{\Lambda} = 2\mu(3\mathbb{E}_1 + \mathbb{E}_3) \tag{A.4}$$

The elements of the Walpole's basis Walpole (1981, 1966) are defined by taking a unit vector perpendicular to the isotropy plane. For the applications in the context of Fourier transforms, this unit vector is equal to a unit vector along a considered wavevector, leading to:

$$\begin{cases} \mathbb{E}_1(\xi) = \frac{1}{2}\mathbf{k}^\perp \otimes \mathbf{k}^\perp \\ \mathbb{E}_2(\xi) = \mathbf{k} \otimes \mathbf{k} \\ \mathbb{E}_3(\xi) = \mathbf{k}^\perp \overline{\underline{\otimes}} \mathbf{k}^\perp - \mathbb{E}_1 \\ \mathbb{E}_4(\xi) = \mathbf{k}^\perp \overline{\underline{\otimes}} \mathbf{k} + \mathbf{k} \overline{\underline{\otimes}} \mathbf{k}^\perp \\ \mathbb{E}_5(\xi) = \mathbf{k} \otimes \mathbf{k}^\perp, \quad \mathbb{E}_6(\xi) = \mathbf{k}^\perp \otimes \mathbf{k} \end{cases} \tag{A.5}$$

where $\mathbf{k}$ and $\mathbf{k}^\perp$ are defined by:

$$\mathbf{k} = \frac{1}{|\xi|^2}\xi \otimes \xi, \quad \mathbf{k}^\perp = \mathbf{i} - \mathbf{k} \tag{A.6}$$

with the condition $|\xi| \neq 0$, and:

$$(\mathbb{A}\overline{\underline{\otimes}}\mathbb{B})_{ijkl} = \frac{1}{2}(A_{ik}B_{jl} + A_{il}B_{jk}) \tag{A.7}$$

Appendix B. Inverse Laplace transform of a polynomial fraction of the Laplace variable

For obtaining time dependent solutions to problems related to viscoelasticity, it is necessary to proceed to the inversion of the Laplace transform. The inverse Laplace transform of a polynomial fraction of the Laplace variable appears in many textbooks. Its expression is recalled below for completeness.

If the Laplace transform of a function $\tilde{f}(s)$ is a polynomial fraction with numerator $P(s)$ and denominator $Q(s)$ such that the degree of $P(s)$ is less than the degree of $Q(s)$ and if $Q(s)$ has n distinct zero α_k, $k = 1, 2, 3, \ldots, n$, the inverse Laplace transform is given by:

$$L^{-1}\left\{\tilde{f}(s)\right\} = L^{-1}\left\{\frac{P(s)}{Q(s)}\right\} = \sum_{k=1}^{n} \frac{P(\alpha_k)}{Q\prime(\alpha_k)} e^{\alpha_k t} \tag{B.1}$$

In the case where the equation $Q(s) = 0$ has a multiple root of order m, while the other roots, $\beta_1, \beta_2, \ldots, \beta n$ are simple, the decomposition into elementary fractions takes the form:

$$\frac{P(s)}{Q(s)} = \sum_{k=1}^{m} \frac{A_k}{(s-\alpha)^{m-k+1}} + \sum_{l=1}^{n} \frac{B_l}{(s-\beta_l)} \tag{B.2}$$

where:

$$\begin{aligned} A_k &= \lim_{s\to\alpha} \frac{1}{(k-1)!} \frac{d^{k-1}}{ds^{k-1}} \left\{(s-\alpha)^m \tilde{f}(s)\right\} \\ B_l &= \frac{P(\beta_l)}{Q\prime(\beta_l)} \end{aligned} \tag{B.3}$$

The inverse Laplace transform is finally given by:

$$f(t) = e^{\alpha t} \sum_{k=1}^{m} A_k \frac{t^{m-k}}{(m-k)!} + \sum_{l=1}^{n} B_l e^{\beta_l t} \tag{B.4}$$

Appendix C. The local strain field within an heterogeneous medium

The local strain field within the heterogeneous medium is obtained classically (Milton, 2002) by solving an integral equation. This appendix recalls how such an integral equation can be obtained.

The solution of the localization problem is obtained by using a "reference medium" whose elasticity tensor is $\mathbb{C}^0$. The cell problem, which is defined by using heterogeneous properties, is replaced by an equivalent problem for a periodic cell containing a homogeneous material having the elastic properties of the "reference medium" and submitted to a suitable field of eigenstrain $\boldsymbol{\epsilon}^L$, or eigenstress $\boldsymbol{\sigma}^L$ (usually named polarization). Obviously these quantities are searched in the following in order to ensure that stresses and strains in the heterogeneous medium are the same as the ones obtained in the reference medium when eigenstrain or eigenstress field are applied. The interest of this new formulation is that the response of the homogeneous "reference" medium, when eigenstrain or eigenstress field are applied, is given explicity by the Green's tensor. Indeed, using equilibrium and compatibility equations lead to the relation between the local strain fields induced by a given eigenstress field by:

$$\widehat{\boldsymbol{\epsilon}}(\xi) = -\widehat{\boldsymbol{\Gamma}}(\xi) : \widehat{\boldsymbol{\sigma}}^L(\xi) \tag{C.1}$$

where Γ is the "strain Green's tensor". A similar relation can be obtained when using the eigenstrain field.

Writing that the application of the eigenstress field must produce the same local strains and stresses as the heterogeneous medium leads to the integral equation of "Lippman–Schwinger–Dyson" type for the strain tensor. To obtain this integral equation, a consistency condition, exactly similar to the ones described in the cases of single inhomogeneity problems can be written as:

$$\mathbb{C}(\mathbf{x}) : \left(\mathbf{E} + \boldsymbol{\epsilon}_{per}(\mathbf{x})\right) = \mathbb{C}^0 : \left(\mathbf{E} + \boldsymbol{\epsilon}_{per}(\mathbf{x})\right) + \boldsymbol{\sigma}^L(\mathbf{x}) \tag{C.2}$$

which produces the eigenstress field related to the "periodic strain tensor":

$$\boldsymbol{\sigma}^L(\mathbf{x}) = \delta\mathbb{C} : \left(\mathbf{E} + \boldsymbol{\epsilon}_{per}(\mathbf{x})\right) \tag{C.3}$$

with:

$$\delta\mathbb{C} = \mathbb{C}(\mathbf{x}) - \mathbb{C}^0 \tag{C.4}$$

The use of the Green's tensor allows to obtain the "periodic strain field" $\boldsymbol{\epsilon}_{per}(\mathbf{x})$ depending on the eigenstress through (C.1). In real space, this relation can be written:

$$\boldsymbol{\epsilon}_{per}(\mathbf{x}) = -\boldsymbol{\Gamma}(\mathbf{x}) * \boldsymbol{\sigma}^L(\mathbf{x}) \tag{C.5}$$

where the star denotes the spatial convolution product. The consistency condition (C.3) and relation (C.5) produce the value of the strain tensor at the microscopic scale induced by the strain tensor at the macroscopic scale:

$$\boldsymbol{\epsilon}(\mathbf{x}) = \mathbf{E} - \boldsymbol{\Gamma}(\mathbf{x}) * (\delta\mathbb{C} : \boldsymbol{\epsilon}(\mathbf{x})) \tag{C.6}$$

This is an integral equation, because it involves an integral containing ϵ through the convolution product. For the case of a matrix-inclusion material, the local elasticity tensor $\mathbb{C}$ can be written as:

$$\mathbb{C}(\mathbf{x}) = \sum_{\alpha} I_\alpha(\mathbf{x})\mathbb{C}_\alpha \tag{C.7}$$

where $\mathbb{C}_\alpha$ is the elasticity tensor of phase α and I_α is the characteristic function describing the volume of phase α. $\alpha \equiv I$ for inclusion and $\alpha \equiv M$ for matrix. Then, (C.4) can be written as:

$$\delta\mathbb{C}(\mathbf{x}) = \left(\mathbb{C}_I - \mathbb{C}^0\right)I_I(\mathbf{x}) + \left(\mathbb{C}_M - \mathbb{C}^0\right)I_M(\mathbf{x}) \tag{C.8}$$

The formal solution of (C.7) produces the Neumann series (17) of Section 2.

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